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In this paper three problems are discussed. First, the resolution of a complex 2-vector M, in space of $2\,p$ dimensions into the sum of p mutually completely perpendicular simple 2-vectors or planes. It is shown that this can always be done and is in general unique. But if M satisfies certain product relations the resolution can be effected in an infinite number of ways. In four dimensions this relation is equivalent to saying that M is what Whitehead [1] calls self-supplementary. In this case the resolution can be effected in ∞^2 different ways.

Second, the application of the preceding to show that a rotation in a space of $2\,p$ dimensions leaves p mutually completely perpendicular planes invariant. In general these are the only invariant planes. But in case the rates of rotation in the p invariant planes are the same or differ only in sign there are an infinite number of invariant planes which can be arranged in sets of p which are mutually completely perpendicular. In 4-space in case the rates of rotation in two completely perpendicular invariant planes have the same magnitude there are ∞^2 invariant planes which are completely perpendicular in pairs.

Third, the consideration of the variety V_p left invariant by all the transformations leaving the same set of p planes invariant. It is found that this variety is of order 2^p. The path curves are curves of constant curvature and first torsion and are geodesics on V_p. The centers of curvature of all the path curves that pass through a given point lie on a sphere of p-dimensions having the given point P and the point of intersection O of the p invariant planes as ends of a diameter. Through each point pass 2^p path curves which are circles having O for center and OP for radius. The variety V_p can be developed on a plane space of p dimensions.

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[1] A treatise on Universal Algebra, page 292.

The only papers that I know of bearing on rotations in hyperspace are given below: [2]

1. **Introduction.** In terms of the Gibbs Vector analysis an infinitesimal rotation in three dimensions can be expressed by the formula [3]

$$r' = r + a \times r \, dt,$$

where a is a vector along the axis of the rotation and r is any vector through the fixed origin. If the rotation is considered as parallel to a fixed plane determined by the vectors b, c then it can be represented by the formula

$$r' = r + (b \times c) \times r \, dt.$$

By the Gibbs definition of the cross product (a vector perpendicular to the plane of the two vectors and of magnitude equal to the product of the lengths of the two vectors into the sine of the angle between them, so arranged that the three vectors $b, c, b \times c$ for a right handed system) this last expression is equivalent to the first. If we wish to extend this to higher dimensions we cannot have a form equivalent to the first since the cross product of two 1-vectors cannot be considered as a 1-vector as, in that case, two 1-vectors do not uniquely determine a 1-vector. We will then have to start with a new set of definitions of the products. It must be kept in mind that we may have vectors of different dimensions as 1-vectors, 2-vectors, 3-vectors, etc., of one, two, three, etc., dimensions. I shall here use the vector analysis already set up by Wilson and Lewis.[4]

The cross product of two-vectors extending from the same origin is defined as the parallelogram defined by the two vectors. The product is then a 2-vector. The magnitude of the product is equal to the area of the parallelogram. Similar definitions are given for the cross

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[2] F. N. Cole: On rotations in space of four dimensions. American Journal, **12**, 1889, page 191.

P. H. Schoute: Le déplacement le plus général dans l'espace à n dimensions. Annales de l'Ecole Polytechnique de Delft, **7**, 1891.

Bemporad: sui gruppi dei movimenti. Annali della r. scuola normale sup. di Pisa, **8**, 1904.

E. E. Levi: Sui gruppi di movimenti. Atti dei Lincei Series 5, **14**, part 1, 1905.

[3] Gibbs-Wilson Vector Analysis, page 99.

[4] Space-time manifold of relativity. The non-euclidean geometry of mechanics and electromagnetics. These Proceedings, **48**, number 11, 1912.

product of more than two 1-vectors. The dot product of two 1-vectors is defined as the projection of one on the other multiplied by the length of the one on which the projection is made. This agrees with the Gibbs definition of the dot product of two 1-vectors. This product is a scalar. The dot product of two vectors of higher dimension is defined as the vector in the larger space perpendicular to the smaller. The magnitude is equal to the magnitude of the projection of the smaller into the magnitude of the larger. If the two vectors are of equal magnitude this product is again a scalar otherwise it is a vector. This definition differs widely from the usual definition of inner product. This product is commutative while the ordinary inner product is not.

We shall choose unit vectors along mutually perpendicular axes for our reference system. Let these reference vectors be $k_1, k_2, \ldots . k_n$. Then the coordinate planes, 3-spaces, 4-spaces, etc. are

$$k_{12} = k_1 \times k_2,\ k_{13} = k_1 \times k_3, \ldots . k_{rs} = k_r \times k_s$$
$$k_{123} = k_1 \times k_2 \times k_3, \ldots . k_{pqr} = \mathrm{k}_p \times k_q \times k_r,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$k_{pq} \ldots . r = k_p \times k_q \times \ldots . \times k_r.$$

The dot product of these unit vectors are as follows:

$$k_i \cdot k_i = 1,\ k_i \cdot k_j = 0,\ (i \neq j),$$
$$k_{ij} \cdot k_{ij} = 1,\ k_{ij} \cdot k_{il} = 0,\ (j \neq l),\ k_{ij} \cdot k_{mn} = 0\ (i, j \neq m, n)$$
$$k_i \cdot k_{ij} = -k_j,\ k_j \cdot k_{ij} = k_i,$$
$$k_i \cdot k_{ijl} = k_{jl},\ k_{ij} \cdot k_{ijl} = k_l \text{ etc.}$$

The dot product of two of these unit vectors vanishes if the smaller is not entirely contained in the larger, that is if a subscript appears in the smaller which does not also appear in the larger. If the dot product vanishes the two vectors are perpendicular. This however does not require complete perpendicularity, that is it requires that one vector in one is perpendicular to the other while complete perpendicularity requires that every vector in one is perpendicular to every vector in the other. To obtain a vector completely perpendicular to a given vector we must resort to the complement, which for unit vectors is defined as the vector obtained by taking the dot product of the given vector with the pseudo scalar.[5] Throughout this paper

5 The pseudo scalar is defined as the cross product of all the 1-vectors arranged so that the value is unity. Thus in 4-space the pseudo scalar is $k_1 \times k_2 \times k_3 \times k_4 = k_{1234}$.

the word perpendicular shall be used to mean complete perpendicularity.

The formular for the reduction of the various products of 1- and 2-vectors are,

(1). $$a\cdot(b\times c) = (a\cdot c)b - (a\cdot b)c$$
(2). $$(a\times b)\cdot C = a\cdot(b\cdot C) = -b\cdot(a\cdot C)$$
(3). $$a\times(b\cdot C) = (a\times C)\cdot b - (a\cdot b)C$$
(4). $$(b\cdot C)\cdot A = -b(C\cdot A) + C\cdot(b\times A)$$

where a, b, c are 1-vectors and A, C are 2-vectors.

Now having these definitions an infinitesimal rotation [6] parallel to a fixed plane M_1 is defined by the equation

$$r' = r + M_1\cdot r\, dt.$$

The length of r' is equal to the length of r. For

$$r'\cdot r' = r\cdot r + 2r\cdot(M_1\cdot r)dt = r\cdot r$$

The product $r\cdot(M_1\cdot r)$ vanishes since $M_1\cdot r$ is defined as a vector in M_1 perpendicular to r. A general rotation can be considered as made up of rotations parallel to a number of independent planes. The equation for such a rotation is

$$r' = r + (M_1 + M_2 + \ldots + M_k)\cdot r\, dt.$$

The sum of k, simple plane vectors is a complex [7] plane vector or 2-vector. Therefore we may write the rotation in the form

(5) $$r' = r + M\cdot r\, dt$$

where M is a complex 2-vector, that is, is not equivalent to a simple 2-vector. The canonical form then which (5) may take depends on the form in which M may be written. Then we shall first show that M can always be resolved into the sum of p mutually perpendicular planes if we are working in a space of $2\ p$ dimensions. We first consider the cases of four dimensions and five dimensions and then generalize to space of any number of dimensions.

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[6] A rotation is defined as a rigid motion leaving one point fixed.

[7] Plane vectors in 4-space, for example, are analogous to lines in 3-space. We know that the sum of two lines is not a line unless the lines intersect. The same is true of plane vectors, their sum is a complex or complex vector unless the two simple plane vectors have a line in common. We shall use complex as equivalent to complex vector.

II.

2. **Complex 2-vectors in 4-space.** Let k_1, k_2, k_3, k_4 be four mutually perpendicular unit vectors. Then any 1-vector can be expressed as a linear function of these four and any 2-vector (simple or complex) can be represented as a linear function of the six coordinate planes $k_{ij} = k_i \times k_j$. Thus

$$(6) \qquad M = a_{12}k_{12} + a_{13}k_{13} + a_{14}k_{14} + a_{23}k_{23} + a_{24}k_{24} + a_{34}k_{34}.$$

From this equation we have at once, *A complex 2-vector can be resolved into the sum of two simple planes* M_1 and M_2, *one passing through an arbitrary 1-vector and the other lying in a 3-space perpendicular to it.* For let

$$M_1 = a_{12}k_{12} + a_{13}k_{13} + a_{14}k_{14}$$

This is a simple plane since it is the sum of three simple 2-vectors having the vector k_1 in common. As we could choose for k_1 a unit vector in any direction, this plane can be made to pass through an arbitrary 1-vector. Then let

$$M_2 = a_{23}k_{23} + a_{24}k_{24} + a_{34}k_{34}.$$

This is a plane since it is the sum of three simple plane vectors which lie in the 3-space determined by k_2, k_3, k_4. This 3-space is evidently perpendicular to k_1 since $k_1 \cdot (k_2 \times k_3 \times k_4) = 0$. (If we did not confine ourselves to a rectangular set of axes this 3-space would not necessarily be perpendicular to k_1). We can then write

$$M = M_1 + M_2.$$

The condition that M be a simple plane vector is

$$M \times M = 0$$

For,

$$M \times M = (M_1 + M_2) \times (M_1 + M_2) = 2M_1 \times M_2.$$

In 4-space the cross product of two 2-vectors is a scalar and if this product vanishes it signifies that M_1 and M_2 lie in a 3-space and consequently $M_1 + M_2$ can be expressed as a simple plane vector.

A complex 2-vector can always be resolved into the sum of two simple

2-vectors, one of which is arbitrary. For, let A be any plane vector, then $M-\lambda A$ will be a simple plane if

$$(M-\lambda A)\times(M-\lambda A) = M\times M - 2\lambda\, M\times A = 0$$

or

$$\lambda = \frac{M\times M}{2M\times A}$$

and we can write

$$M = \frac{M\times M}{2M\times A}A + (M - \frac{M\times M}{2M\times A}A)$$

We shall now show that the complex 2-vector can always be resolved in at least one way into the sum of two perpendicular planes. Let

$$M = m_1 M_1 + m_2 M_2 \tag{8}$$

where M_1 and M_2 are completely perpendicular unit planes and m_1 and m_2 numbers. Then indicating $(M\times M)\cdot M$ by A we can write

$$A = (M\times M)\cdot M = 2m_1m_2(M_1\times M_2)\cdot M = 2m_1m_2(m_2M_1+m_1M_2). \tag{9}$$

Solving (8) and (9) for M_1 and M_2 we have

$$\begin{aligned} M_1 &= \frac{2m_2^2M - A}{2m_2(m_2^2 - m_1^2)} \\ M_2 &= \frac{2m_1^2M - A}{2m_1(m_1^2 - m_2^2)} \end{aligned} \tag{10}$$

The values of m_1 and m_2 can be computed from the relations

$$\begin{aligned} M\cdot M &= m_1^2 + m_2^2 \\ A\cdot M &= 4m_1^2m_2^2 \end{aligned}$$

From which we get

$$\begin{aligned} m_1 &= \tfrac{1}{2}\sqrt{M\cdot M + \sqrt{M\cdot A}} + \tfrac{1}{2}\sqrt{M\cdot M - \sqrt{M\cdot A}} \\ m_2 &= \tfrac{1}{2}\sqrt{M\cdot M + \sqrt{M\cdot A}} - \tfrac{1}{2}\sqrt{M\cdot M - \sqrt{M\cdot A}}\ . \end{aligned}$$

If $m_1 \neq \pm m_2$ the solution (10) is unique.[8] If $m_1 = \pm m_2$ the solutions become infinite. From (9) we see at once that this relation means

[8] For the non-euclidean geometry considered by Wilson and Lewis this resolution was unique since in that case the denominators contained the factor $m_1^2 + m_2^2$ instead of $m_1^2 - m_2^2$.

that $A = \pm M$ or the complement of M is $\pm M$. (From the definition of complement, the complement of M is $m_2M_1 + m_1M_2$). If we indicate complement by * we have the relations for 4-space.

$$k^*_{12} = k_{34},\ k^*_{13} = k_{42},\ k^*_{14} = k_{23},\ k^*_{23} = k_{14},\ k^*_{24} = k_{31},\ k^*_{34} = k_{12}.$$

Then if a complex is such that $M^* = \pm M$ it can be written in the form

$$M = (a_{12}k_{12} + a_{13}k_{13} + a_{14}k_{14}) \pm (a_{13}k_{42} + a_{14}k_{23} + a_{12}k_{34}).$$

The expressions in brackets are completely perpendicular planes and in this case we have a resolution into completely perpendicular planes. Hence: *A complex 2-vector can always be resolved into the sum of two completely perpendicular planes.*

By a proper choice of axes M can be written in the form

$$M = m_1k_{12} + m_2k_{34}$$

and if $M^* = \pm M$ the above resolution is not unique. We shall now proceed to investigate this case. We saw that a complex could always be resolved into the sum of two planes one of which was arbitrary. Let the arbitrary plane be X and write

$$M = X + K.$$

If r is any vector in X, consider the transformation

$$r' = r \cdot M = r \cdot X + r \cdot K.$$

$r \cdot X$ is a vector in X and if r' also lies in X, $r \cdot K$ must vanish since X and K have no vector in common. But since r was any vector in X the vanishing of $r \cdot K$ for all values of r means that K must be completely perpendicular to X. Hence the resolution of M into the sum of two perpendicular planes resolves itself into finding the planes left invariant by the transformation $r' = r \cdot M$. We shall therefore find the planes left invariant by this transformation.

The invariants of this transformation are more easily found by expressing it in the form of a dyadic. From formula (1) we have

$$r \cdot (k_{ij}) = (r \cdot k_j)k_i - (r \cdot k_i)k_j$$

which can be written as the dot product of r into the dyadic $k_jk_i - k_ik_j$. The transformation

$$r' = r \cdot M$$

then takes the form

(12) $$r' = r\cdot[m_1(k_2k_1 - k_1k_2) + m_2(k_4k_3 - k_3k_4)] = r\cdot\Phi$$

where $\Phi = m_1(k_2k_1 - k_1k_2) + m_2(k_4k_3 - k_3k_4)$.

If $m_1 \neq \pm m_2$ the only 1-vectors left invariant[9] are

$$k_1 \pm ik_2, \quad k_3 \pm ik_4.$$

For if
$$r = \lambda_1k_1 + \lambda_2k_2 + \lambda_3k_3 + \lambda_4k_4,$$
$$r' = r\cdot\Phi = -m_1\lambda_1k_2 + m_1\lambda_2k_1 - m_2\lambda_3k_4 + m_2\lambda_4k_3$$

and if this is to be equal to μr

(13) $$m_1\lambda_2 = \mu\lambda_1, - m_1\lambda_1 = \mu\lambda_2, m_2\lambda_4 = \mu\lambda_3, - m_2\lambda_3 = \mu\lambda_4$$

which can be satisfied only for the values

$$\lambda_2 = \pm i\lambda_1, \lambda_3 = 0, \lambda_4 = 0; \quad \lambda_1 = \lambda_2 = 0, \lambda_4 = \pm i\lambda_3$$

If $m_1 = m_2$ we see that (13) is satisfied by any vector of the pencils

$$k_1 + ik_2 + \lambda(k_3 + ik_4), \quad k_1 - ik_2 + \lambda(k_3 - ik_4),$$

that is these vectors are left invariant for all values of λ. If $m_1 = -m_2$, the pencils

$$k_1 + ik_2 + \lambda_3(k_3 - ik_3), \quad k_1 - ik_2 + \lambda(k_3 + ik_4)$$

are left invariant for all values of λ.

If we apply the transformation twice, that is

$$r' = r\cdot\Phi, \quad r'' = r'\cdot\Phi = (r\cdot\Phi)\cdot\Phi$$

assuming the above form for r we have

$$r'' = -m_1^2(\lambda_1k_1 + \lambda_2k_2) - m_2^2(\lambda_3k_3 + \lambda_4k_4).$$

If $m_1 = \pm m_2$ r'' is a multiple of r and if $m_1 = \pm m_2 = 1$, the transformation repeated twice is a reflection through the origin and of course repeated four times is the identical transformation. These are the only conditions under which the transformation will close. Hence *the necessary and sufficient condition that* (12) *be of finite order is* $M_1 = \pm m_2$ *or* $M^* = \pm M$. In this case there are two pencils of invariant vectors while in the general case only four vectors are left invariant.

In order to find the 2-vectors left invariant by the transformation

[9] Invariant here means that $r' = \mu r$.

we will express Φ in terms of planes (2-vectors). This is done by means of the double product of Gibbs.[10] If $\Phi = \Sigma\ k_i l_i$ is a dyadic which transforms 1-vectors into 1-vectors then $\frac{1}{2}\,\Phi{}^{\times}_{\times}\Phi = \frac{1}{2}\,\Sigma\ (k_i{\times}k_j)(l_i{\times}l_j)$ represents the same transformation expressed in plane coordinates, that is a transformation which transforms planes into planes. To show this let r and s be two 1-vectors. Then the plane $r{\times}s$ is transformed by Φ into $(r\cdot\Phi){\times}(s\cdot\Phi)$ which can be written

$$\begin{aligned}
r'{\times}s' &= (r\cdot\Phi){\times}(s\cdot\Phi) = \tfrac{1}{2}[(r\cdot\Phi){\times}(s\cdot\Phi) - (s\cdot\Phi){\times}(r\cdot\Phi)] \\
&= \tfrac{1}{2}\{[\Sigma(r\cdot k_i)l_i]{\times}[\Sigma(s\cdot k_j)l_j] - [\Sigma(s\cdot k_i)l_i]{\times}[\Sigma(r\cdot k_j)l_j]\} \\
&= \tfrac{1}{2}\{\Sigma[(r\cdot k_i)(s\cdot k_j) - (s\cdot k_i)(r\cdot k_j)](l_i{\times}l_j) \\
&= \tfrac{1}{2}\Sigma[(r{\times}s)\cdot(k_i{\times}l_j)](l_i{\times}l_j) \\
&= \tfrac{1}{2}\ (r{\times}s)\cdot[\Phi{}^{\times}_{\times}\Phi].
\end{aligned}$$

If Φ is the dyadic used in (12)

$$\Phi = m_1(k_2k_1 - k_1k_2) + m_2(k_4k_3 - k_3k_4). \tag{14}$$

Then

$$\Psi = \tfrac{1}{2}\Phi{}^{\times}_{\times}\Phi = m_1^2k_{12}k_{12} + m_2^2k_{34}k_{34} + m_1m_2(k_{13}k_{24} + k_{24}k_{13} - k_{14}k_{23} - k_{23}k_{14}). \tag{15}$$

If

$$P = \Sigma a_{ij}\, k_{ij}$$

be any complex 2-vector

$$P\cdot\Psi = m_1^2a_{12}k_{12} + m_1m_2a_{24}k_{13} - m_1m_2a_{23}k_{14} - m_1m_2a_{14}k_{23} + m_1m_2a_{13}k_{24} + m_2^2a_{34}k_{34}$$

and if $P\cdot\Psi = \lambda P$ we see that this is satisfied by the planes k_{12} and k_{34} and by the complexes

$$a_{13}(k_{13} + k_{24}) + a_{14}(k_{14} - k_{23});\ a_{13}(k_{13} - k_{24}) + a_{14}(k_{14} + k_{23}).$$

These complexes satisfy the condition for all values of a_{13} and a_{24}. The invariant planes will then be k_{12} and k_{34} and the planes belonging to either of these pencils of complexes. These last named planes are obtained from the values of $a_{13} : a_{14}$ satisfying either of the relations

$$\begin{aligned}
&[a_{13}(k_{13} + k_{24}) + a_{14}(k_{14} - k_{23})]{\times}[a_{13}(k_{13} + k_{24}) + a_{14}(k_{14} - k_{23})] = 0 \\
&[a_{13}(k_{13} - \mathrm{k}_{24}) + a_{14}(k_{14} + k_{23})]{\times}[a_{13}(k_{13} - k_{24}) + a_{14}(k_{14} + k_{23})] = 0
\end{aligned}$$

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10 See Phillips and Moore, Dyadics occurring in point space of three dimensions. These Proceedings, vol. 53.

These give the values $a_{14} = \pm i a_{13}$. Hence the invariant planes are

(16) $\qquad (k_{13} + k_{24}) \pm i(k_{14} - k_{23}), \quad (k_{13} - k_{24}) \pm i(k_{14} + k_{23}).$

Now if $m_1 = \pm m_2$

(17) $\quad \Psi_1 = m_1^2[(k_{12}k_{12} + k_{34}k_{34}) \pm (k_{13}k_{42} + k_{42}k_{13} + k_{14}k_{23} + k_{23}k_{14})].$

Hence the complex

(18) $\qquad P = a_{12}k_{12} + a_{13}(k_{13} \pm k_{42}) + a_{14}(k_{14} \pm k_{23}) + a_{34}k_{34}.$

(Both positive or both negative signs are to be taken together), is left invariant for all values of $a_{12}, a_{13}, a_{14}, a_{34}$. The planes which belong to this system of complexes are determined by the relation

$$P \times P = a_{12}a_{34} - a_{13}^2 - a_{14}^2 = 0.$$

Substituting in (18) we have for the invariant planes

$$K_1 = (a_{13}^2 + a_{14}^2)k_{12} + a_{13}a_{34}(k_{13} \pm k_{42}) + a_{14}a_{34}(k_{14} \pm k_{23}) + a_{34}^2k_{34}$$

The plane

$$K_2 = a_{34}^2k_{12} - a_{13}a_{34}(k_{13} \pm k_{42}) - a_{14}a_{13}(k_{14} \pm k_{23}) + (a_{13}^2 + a_{14}^2)k_{34}$$

also belongs to (18) and is completely perpendicular to K_1. Hence *the planes left invariant by the transformation $P' = P \cdot \Psi_1$ form a two parameter family and are completely perpendicular in pairs.* The planes belong to a three parameter linear system of complexes and so must cut two fixed planes. It is not difficult to see that these are the planes (16). The first pair when $m_1 = m_2$ and the second pair when $m_1 = -m_2$. We can now write

$$M = m_1(k_{12} + k_{34}) = \frac{m_1}{a_{34}^2 + a_{13}^2 + a_{14}^2}(K_1 + K_2).$$

Hence the theorem may be stated: *Any complex 2-vector can be resolved into the sum of two completely perpendicular planes. If $M^* \neq \pm M$ the resolution is unique. If $M^* = \pm M$ the resolution can be made in ∞^2 ways.*

3. **Complex 2-vectors in 5-space.** In 5-space a complex 2-vector M can be resolved into the sum of two simple plane vectors. For,

if we express the complex in terms of the unit coordinate planes we have

$$M = \sum_{i,j=1}^{5} a_{ij}k_{ij} = \sum_{1}^{5} a_{5j}k_{5j} + \sum_{i,j=1}^{4} a_{ij}k_{ij}.$$

The sum $\Sigma a_{5j} k_{5j}$ represents a simple plane vector since each term in it contains the vector k_5. The sum $\sum_1^4 a_{ij}k_{ij}$ represents a complex 2-vector lying in the 4-space determined by k_1, k_2, k_3, k_4 and hence can be expressed as the sum of two plane vectors in this 4-space, one of which is arbitrary. The plane $A = \Sigma a_5 k_{5j}$ will cut the 4-space $k_1 \times k_2 \times k_3 \times k_4$ in a 1-vector. Now resolve $\sum_1^4 a_{i5}k_{ij}$ into the sum of two simple plane, $B + C$ and choose B so that it will contain the vector in which A cuts the 4-space. Since A and B have a vector in common, $A + B$ will be a simple plane vector D and we have $M = C + D$ where C and D are simple planes. Neither of these planes can be chosen arbitrarily as was the case in 4-space. It follows from the fact that a complex vector is always expressible as the sum of two plane vectors that it must necessarily lie in a 4-space. This 4-space is the same no matter how the complex M is expressed as the sum of two simple planes. For, if

$$M = M_1 + M_2$$

where M_1 and M_2 are simple planes, then

$$M \times M = 2M_1 \times M_2$$

But $M \times M$ is the same however M is expressed hence the 4-space $M_1 \times M_2$ must be the same however M is expressed as the sum of two plane vectors. Since a plane vector in 4-space can be resolved into the sum of two completely perpendicular 2-vectors the same holds true for complex 2-vectors in 5-space. Just as in 4-space if

$$M = \lambda(M_1 + M_2) \text{ or } M = \lambda(M_1 - M_2)$$

where M_1 and M_2 are completely perpendicular unit planes, the resolution into the sum of perpendicular planes can be effected in ∞^2 ways.

Besides leaving the four imaginary vectors found in 4-space invariant, the transformation $r \cdot M$ in 5-space annihilates the real vector perpendicular to $M \times M$. The products of this transformation then with itself can never be equal to the identical transformation. If

$M = M_1 \pm M_2$ where M_1 and M_2 are unit planes, then the transformation repeated four times will be the identical transformation for vectors in $M \times M$.

4. **Complex 2-vectors in space of 2p dimensions.** We shall first show that if a complex 2-vector in a space of $2p$ dimensions can be resolved into the sum of p independent simple plane vectors one of which passes through an arbitrary 1-vector then a complex 2-vector in a space of $2p+1$ dimensions can also be resolved into the sum of p independent simple plane vectors but in this case no one of the simple 2-vectors can be made to pass through an arbitrary 1-vector. To show this it is only necessary to write the complex in $2p+1$ dimensions in terms of unit coordinate planes

$$M = \sum_{1}^{2p+1} a_{1j}k_{1j} + \sum_{2}^{2p+1} a_{ij}k_{ij}.$$

The first sum represents a simple plane vector since each term contains the vector k_1. The second sum represents a complex 2-vector lying in the space of $2p$ dimensions determined by the vectors $k_2, k_3, \ldots k_{2p+1}$ and therefore by the above assumption can be expressed as the sum of p simple plane vectors one of which, A say, passes through the vector in which the plane $B = \sum_{1}^{2p+1} a_{1j}k_{1j}$ cuts the space in which the complex $\sum_{2}^{2p+1} a_{ij}k_{ij}$ lies. Then since A and B have a vector in common, their sum can be expressed as a simple plane vector. The complex M is then expressed as the sum of p independent planes. But p independent simple plane vectors determine a space of $2p$ dimensions and this space of $2p$ dimensions is the same no matter how M is expressed. For if

$$M = M_1 + M_2 + \ldots + M_p.$$

Then

$$M \times M \times M \ldots p \text{ factors} = p! \quad M_1 \times M_2 \times \ldots \times M_p;$$

and since the first member is independent of how the complex is expressed as the sum of p independent planes, the second must be.

Again, if a complex 2-vector in a space of $2p$ dimensions can be resolved into the sum of p independent simple plane vectors one of which passes through an arbitrary 1-vector, then a complex 2-vector

in a space of $2p + 2$ dimensions can be resolved into the sum of $p + 1$ independent simple plane vectors one of which passes through an arbitrary 1-vector. For let the complex be expressed in terms of the unit coordinate planes

$$M = \sum_{1}^{2p+2} a_{1j}k_{1j} + \sum_{2}^{2p+2} a_{ij}k_{ij}.$$

The first sum is a simple 2-vector passing through k_1 which can be chosen arbitrarily. The second sum is a complex 2-vector in a space of $2p + 1$ dimensions and consequently can be expressed as the sum of p independent simple plane vectors. Hence the whole complex can be expressed as the sum of $p + 1$ independent planes. We have seen that a complex 2-vector in 4-space can be resolved into the sum of two independent simple plane vectors one of which can be chosen arbitrarily and therefore by induction we have: *a complex 2-vector in $2p$ or $2p + 1$ dimensions can always be resolved into the sum of p independent simple plane vectors.*

The condition that a 2-vector in a space of $2p$ dimensions be simple, is

$$M \times M = 0.$$

For if this condition is satisfied then the following relations are satisfied owing to the associative character of the multiplication when the order of the whole product is equal to or less than $2p$

$$\begin{gathered} M \times M \times M = 0 \\ M \times M \times M \times M = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ M \times M \times M \times \ldots \times p \text{ factors} = 0. \end{gathered}$$

The last one shows that the complex must lie in a space of lower dimensions and therefore can be expressed as the sum of $p - 1$ simple plane vectors. By the same argument it can be expressed as the sum of $p - 2$ simple plane vectors and so on until finally it can be expressed as a single simple plane vector. If M is a simple plane, $M \times M = 0$. Hence this is both a necessary and a sufficient condition that a 2-vector be simple.

We shall next show that a complex 2-vector in a space of $2p$ dimensions can be resolved into the sum of p mutually perpendicular simple planes. Let

$$M = \sum_{1}^{p} m_i M_i$$

when M_i are mutually perpendicular unit simple planes and m_i numbers. We can then derive $p-1$ complex 2-vectors as follows

$$
\begin{aligned}
&A = (M\times M)\cdot M = 2\sum_{1}^{p} m_i m_j^2 M_i \qquad i \neq j \\
&B = (M\times M\times M)\cdot(M\times M) = 6\sum_{1}^{p} m_i m_j^2 m_k^2 M_i \qquad i \neq j \neq k. \\
(20)\quad &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&P = (M\times M\times M\ldots p \text{ factors})\cdot(M\times M\times M\ldots(p\text{-}1) \text{ factors}) \\
&\quad = p!\,m_1 m_2\ldots m_p \Sigma m_1 m_2 \ldots m_{i-1} m_{i+1}\ldots m_p M_i.
\end{aligned}
$$

We have then all together p equations to solve for the p plane M_1, $M_2, \ldots M_p$. and the solution will be unique unless these equations prove not to be linearly independent, that is unless the determinant Δ of the system vanishes. If we observe how the columns of this determinant are made up we see that it contains each of the m's as a factor. Also if $m_i = \pm m_j$, $\Delta = 0$. Therefore we can write Δ in the factored form

$$\Delta = \cdot p!(m_1 m_2 \ldots m_p)\pi(m_i^2 - m_j^2).$$

Now if $m_1 = \pm m_2$ say, and the other m's are all different then a value of λ can be found so that

$$A + \lambda M$$

will lie in a space of $2p-4$ dimensions and by the above argument this new complex can be resolved into the sum of $p-2$ mutually perpendicular plane vectors. The remaining 4-space will be completely perpendicular to the space in which $A + \lambda M$ lies and that part of M lying in it can be resolved into the sum of two perpendicular planes in ∞^2 ways. Hence the whole complex can be resolved into the sum of p mutually perpendicular planes in ∞^2 ways. The same argument of course applies to any pair of equal roots. If $m_1 = \pm m_2 = \pm m_3$ the other m's being all different and different from m_1; a value of λ can be found so that $A + \lambda M$ will lie in a space of $2p-6$ dimensions and the part of M lying in this space can be resolved into $p-3$ mutually perpendicular planes. The whole complex can then be resolved into the sum of p mutually perpendicular planes provided the complex 2-vector in space of six dimensions such that $(M\times M)\cdot M = \lambda M$ can be resolved into the sum of three mutually perpendicular planes. Continuing the argument we see that the resolution is always possible provided that in a space of $2m$ dimensions in which

$$(M\times M)\cdot M = \lambda M$$

can be resolved into the sum of mutually perpendicular planes. We will now show that this resolution is always possible.

If the transformation

$$r' = r \cdot M = r \cdot (X + K)$$

where X is a simple plane and r any vector in X, always gives a vector r' in X then $r \cdot K = 0$ for every r in X. That is X is completely perpendicular to K. Then the above resolution depends upon whether we can find a plane left invariant by the transformation $r' = r \cdot M$ where M is any complex 2-vector. Let

$$M = \sum_{1}^{2m} a_{ij} k_{ij} \tag{19}$$

and

$$r = \sum_{1}^{2m} b_i k_i.$$

Then if r is an invariant vector we have

$$r \cdot M = \mu r$$

which is equivalent to the set of equations

$$\begin{aligned} \mu b_1 &= \sum_{1}^{2m} a_{1j} b_j \\ \mu b_2 &= \sum_{1}^{2m} a_{2j} b_j \\ &\cdots\cdots\cdots\cdots \\ \mu b_{2m} &= \sum_{1}^{2m} a_{2m\,j} b_j \end{aligned} \tag{21}$$

where the b's are to be determined. The coefficients a_{ij} are seen to satisfy the relations

$$a_{ii} = 0, \quad a_{ij} = -a_{ji}, \quad i \neq j.$$

Equations (20) being homogeneous will have a solution provided the determinant of the system vanishes. This determinant is seen to be a skew determinant with each term in the principal diagonal equal to μ. From the theory of such determinants [11] it is known that it can be expanded in powers of the diagonal terms. The coefficients of the

11 Hanus, Elements of determinants. Ginn & Co., page 152.

various powers of μ are sums of principal minors of the determinant in which the μ's are replaced by zeros. That is the determinant can be expanded in the form

$$\mu^{2m} + A_1\mu^{2m-1} + A_2\mu^{2m-2} + \ldots. + A_{2m} = 0 \tag{22}$$

where A_1 is the sum of the first minors in which μ has been replaced by zero. These minors are then skew symmetric determinants of odd order and therefore vanish. For the same reason all the A's with odd subscripts vanish. The coefficient A_2 is the sum of all the second principal minors with μ replaced by zero. This is a skew symmetric determinant of even order and therefore can be expressed as a sum of squares, consequently it is positive. The same is true of all A's with even subscripts. Then (22) has only even powers of μ and all the coefficients are positive. Therefore the roots appear in conjugate imaginary pairs. This means for M real the invariant vectors appear in conjugate imaginary pairs. But two conjugate imaginary vectors determine a real plane. Hence we have shown that there are always m invariant real planes. If equations (21) are not all independent there will be an infinite number of values of μ and consequently an infinite number of invariant planes. This corresponds to the case $(M{\times}M)\cdot M = \lambda M$.

Now having determined that in every case there is at least one invariant simple plane, A say, we can determine λ so that $M - \lambda A$ will be a complex vector lying in a space of $2m - 2$ dimensions which must be completely perpendicular to A. To determine λ we have the relation

$$\begin{aligned}(M - \lambda A)\times(M - \lambda A)\times\ldots m \text{ factors} = 0\\ = M\times M\times\ldots m \text{ factors} - m\lambda(A\times M\times M\ldots(m-1)\text{ factors}).\end{aligned}$$

From which

$$\lambda = \frac{M^m}{{}_mA\times M^{m-1}}$$

when the exponents indicate cross multiplication. The same reasoning as used before will show that the space in which $M - \lambda A$ lies is completely perpendicular to A. Thus we have established the theorem: *A complex 2-vector in a space of 2p dimensions can always be resolved into the sum of p mutually perpendicular planes. If the set of 2-vectors A, B,— P of (20) are not independent this resolution is not unique.*

If the 2-vectors $A, B, C \ldots P$ are all multiples of M_1 by a proper choice of axes the complex can be written in the simple form

$$M = m_1(k_{12} \pm k_{34} \pm k_{56} \pm \ldots \pm k_{2p-1,\,2p}).$$

We will confine our attention to the single case

$$M = m_1(k_{12} + k_{34} + \ldots k_{2p-1,\,2p}) \tag{23}$$

and the other combinations of sign can be disposed of in a similar manner. The dyadic which represents the transformation $r' = r \cdot M$ is

$$\Phi = m_1(k_2k_1 - k_1k_2 + k_4k_3 - k_3k_4 + \ldots + k_{2p}k_{2p-1} - k_{2p-1}k_{2p}) \tag{24}$$

and the same transformation expressed in plane coordinates is

$$\Psi = \tfrac{1}{2}\Phi\overset{\times}{\times}\Phi,$$

which can be proved by the same method used for the case of 4-space. Similarly it can be shown that the complex

$$\begin{aligned} C = {} & a_{12}k_{12} + a_{34}k_{34} + a_{56}k_{56} + \ldots a_{2p-1,2p}k_{2p-1,2p} \\ & + a_{13}(k_{13} - k_{42}) + a_{14}(k_{14} - k_{23}) + a_{15}(k_{15} - k_{62}) + \ldots \\ & + a_{35}(k_{35} - k_{46}) + a_{36}(k_{36} - k_{45}) + \ldots . \end{aligned} \tag{25}$$

is left invariant by the transformation $C \cdot \Psi$ where the coefficients in C are entirely arbitrary. The complex will be a simple plane if $C \times C = 0$. This gives for the invariant planes, putting $a_{12} = 1$, for convenience.

$$\begin{aligned} A = {} & k_{12} + (a_{13}^2 + a_{14}^2)k_{34} + (a_{15}^2 + a_{16}^2)k_{56} + \ldots \\ & \qquad + (a_{1,2p-1}^2 + a_{1,2p}^2)k_{2p-1,2p} \\ & + a_{13}(k_{13} - k_{42}) + a_{14}(k_{14} - k_{25}) + a_{15}(k_{15} - k_{26}) + \ldots \\ & + (a_{14}a_{15} - a_{13}a_{16})(k_{35} - k_{46}) + (a_{13}a_{15} - a_{14}a_{16})(k_{36} - k_{45}) + \ldots \\ & + (a_{13}a_{17} + a_{14}a_{18})(k_{38} - k_{47}) + \ldots \end{aligned} \tag{26}$$

Hence ∞^{2p-2} simple planes are left invariant. Then in resolving this complex into the sum of p similarly perpendicular simple planes the first plane can be chosen in ∞^{2p-2} different ways, the second in ∞^{2p-4} ways and so on. Hence the resolution can be effected in $\infty^{2(p-1)!}$ different ways.

By proper choice of axes the general complex can be put in the form

$$M = m_{12}k_{12} + m_{34}k_{34} + \ldots + m_{2p-1,2p}k_{2p-1,2p}. \tag{27}$$

Then the dyadic which represents the transformation $r' = r \cdot M$ is

$$(28) \quad \Phi = m_{12}(k_2k_1 - k_1k_2) + m_{34}(k_4k_3 - k_3k_4) + \ldots] \\ + m_{2p-1,2p}(k_{2p}k_{2p-1} - k_{2p-2}k_{2p})$$

If the transformation is applied twice to the vector $r = \pm b_ik_i$ we have

$$\begin{aligned} r'' &= r \cdot \Phi = (r \cdot \Phi) \cdot \Phi^1 = r \cdot (\Phi \cdot \Phi) \\ &= - [m_{12}^2(k_1k_1 + k_2k_2) + m_{34}^2(k_3k_3 + k_4k_4) + \ldots \\ &\qquad + m_{2p-1,2p}(k_{2p-1}k_{2p-1} + k_{2p}k_{2p})] \end{aligned}$$

Then if Φ and r are real and $m_{12} = \pm m_{34} = \ldots = \pm m_{2p-1,2}p = 1$ we return to r after repeating the transformation four times. Hence *if the complex consists of the sum of p mutually perpendicular unit planes, the transformation $r' = r \cdot M$ will be of order 4.*

5. **The Hamilton-Cayley equation.** From equation (12) we saw that the transformation $r' = r \cdot M$ in 4-space depends on two parameters m_1 and m_2 and to show the relation of these to the Hamilton-Cayley identical equation which Φ must satisfy we will change the reference system to that of the invariant elements. We saw that the invariant vectors were

$$k_1 \pm ik_2, \quad k_3 \pm ik_4.$$

Now put

$$r_1 = \frac{1}{\sqrt{2}}((k_1 + ik_2), \quad r_2 = \frac{1}{\sqrt{2}}(k_1 - ik_2)$$

$$r_3 = \frac{1}{\sqrt{2}}(k_3 + ik_4), \quad r_4 = \frac{1}{\sqrt{2}}(k_3 - ik_4),$$

Then

$$k_1 = \frac{i}{\sqrt{2}}(r_1 + r_2), \quad k_2 = -\frac{i}{\sqrt{2}}(r_1 - r_2),$$

$$k_3 = \frac{i}{\sqrt{2}}(r_3 + r_4), \quad k_4 = -\frac{i}{\sqrt{2}}(r_3 - r_4).$$

The dyadic Φ then becomes

$$(29) \qquad \Phi = i[m_1(r_2r_1 - r_1r_2) + m_2(r_4r_3 - r_3r_4)].$$

The multiplication table for the r's is as follows

$$r_1, \cdot r = r_2 \cdot r_2 = r_3 \cdot r_3 = r_4 \cdot r_4 = 0,$$
$$r_1 \cdot r_3 = r_1 \cdot r_4 = r_2 \cdot r_3 = r_2 \cdot r_4 = 0,$$
$$r_1 \cdot r_2 = r_3 \cdot r_4 = 1.$$

The idemfactor I_1 becomes

$$I_1 = r_1r_2 + r_2r_1 + r_3r_4 + r_4r_3. \tag{30}$$

The Hamilton-Cayley equation then becomes

$$(\Phi - im_1I_1)(\Phi + im_1I_1)(\Phi - im_2I_1)(\Phi + im_2I_1) = (\Phi^2 + m_1^2I_1)(\Phi^2 + m_2^2I_1) = 0. \tag{31}$$

If $m_1 = \pm m_2$

$$\Phi = im_1[(r_2r_1 - r_1r_2) \pm (r_4r_3 - r_3r_4)] \tag{29'}$$

from which we see at once that the vectors

$$ar_1 \pm br_3, \quad ar_2 \pm br_4$$

are left invariant (or multiplied by constants only) for all values of a and b. The Hamilton-Cayley equation in this case becomes

$$(\Phi^2 + m_1^2I_1) = 0$$

In terms of this new reference system the dyadic Φ expressed in plane coordinates becomes

$$\Psi = \tfrac{1}{2}\Phi{}^\times_\times\Phi = -[m_1^2r_{12}r_{12} + m_2^2r_{34}r_{34} + m_1m_2(r_{13}r_{24} + r_{42}r_{13} - r_{23}r_{14} - r_{14}r_{23})] \tag{32}$$

The multiplication table for the coordinate planes is

$$r_{12} \cdot r_{12} = r_{34} \cdot r_{34} = -1,$$
$$r_{13} \cdot r_{24} = r_{14} \cdot r_{23} = 1,$$

and all the other products are zero. The idemfactor is

$$I_2 = \tfrac{1}{2}I_1{}^\times_\times I_1 = -r_{12}r_{12} - r_{34}r_{34} + r_{13}r_{24} + r_{24}r_{13} + r_{14}r_{23} + r_{23}r_{14}$$

and the Hamilton-Cayley equation is

$$(\Psi - m_1^2I_2)(\Psi - m_2^2I_2)(\Psi^2 - m_1^2m_2^2I_2) = 0,$$

and if $m_1 = \pm m_2$ the equation is

$$(\Psi - m_1^2I_2)(\Psi + m_1^2I_2) = 0.$$

From (32) we see that the complex 2-vectors

$$ar_{13} + br_{24}, \quad ar_{14} + br_{23}$$

are left invariant for all values of a and b. The only planes belonging to this system are r_{13}, r_{24}, r_{14}, r_{23}. If $m_1 = \pm m_2$ we see at once that the complex

$$ar_{12} + br_{34} + cr_{13} + dr_{24},$$

or

$$ar_{12} + br_{34} + cr_{14} + dr_{23}$$

is left invariant and the planes belonging to this system are the invariant planes discussed before.

Choosing the invariant vectors for the coordinate system in space of $2p$ dimensions and proceed as above we at once arrive at the Hamilton-Cayley equation

$$(\Phi^2 + m_1^2 I_1)(\Phi^2 + m_2^2 I_1)\ldots(\Phi^2 + m_p^2 I_1) = 0$$

and if $m_1 = \pm m_2 = \pm m_3 \ldots = \pm m_p$ this equation becomes

$$\Phi^2 + m_1^2 I_1 = 0.$$

The equation which $\Psi = \frac{1}{2}\Phi{}^{\times}_{\times}\Phi$ satisfies follows in a similar manner.

III.

6. **Rotations in 4-space.** We saw that an infinitesimal rotation could be represented by

$$(5) \qquad r' = r + M \cdot r \, dt.$$

If M is a simple plane vector M_1 say, and if r is perpendicular to M_1, then since $M_1 \cdot r = 0$, the vector r is left absolutely fixed, and therefore the plane completely perpendicular to M_1 is left absolutely fixed. Also if r lies in M_1 it is evident that r' will likewise lie in M_1 and therefore M_1 is left invariant but not point for point. If in this rotation we take M_1 as a unit plane and write (5) in the form

$$r' = r + m_1 M_1 \cdot r \, dt$$

the constant m_1 measures the rate of rotation in the plane M_1. For if r lies in M_1

$$\frac{r' - r}{dt} = \frac{dr}{dt} = m_1 M_1 \cdot r_1$$

Since r does not change in magnitude as it rotates, the magnitude of $\frac{dr}{dt}$ divided by $\sqrt{r \cdot r}$ will be the rate at which r is turning in the plane M_1. That is

$$\sqrt{\left(\frac{dr}{dt}\right)^2} = \sqrt{m_1^2(M_1 \cdot r)^2} = m_1\sqrt{r \cdot r}$$

M_1 being a unit-plane $M_1 \cdot r$ has the same magnitude as r if r lies in M_1. Therefore

$$m_1 = \frac{\sqrt{\left(\frac{dr}{dt}\right)^2}}{\sqrt{r \cdot r}}.$$

which shows that m_1 measures the rate of rotation in M_1.

If in (5) M is a complex 2-vector it can be resolved into the sum of two perpendicular planes and can then be written in the form

$$r' = r + (m_1M_1 + m_2M_2) \cdot r\, dt \tag{33}$$

where M_1 and M_2 are unit planes. In this case the motion consists of a double rotation, one parallel to the simple plane M_1 and the other parallel to the simple plane M_2. The same argument as used above will show that m_1 measures the rate of rotation in M_1 and m_2 the rate of rotation in M_2. If r lies in M_1, then

$$r' = r + M_1 \cdot r\, dt$$

and the same argument used above shows that M_1 is left invariant. The same reasoning also shows that M_2 is left invariant.

In order to exhibit the whole list of invariants we will represent the transformation (33) as a dyadic, choosing the reference system so that

$$M = m_1k_{12} + m_2k_{34}.$$

The same argument as used in §2 shows that the dyadic sought is

$$\Psi = I_1 + [m_1(k_2k_1 - k_1k_2) + m_2(k_4k_3 - k_3k_4)dt = I_1 + \Phi\, dt \tag{34}$$

and (33) then becomes

$$r' = r \cdot \Psi. \tag{35}$$

If the vector

$$r = \sum_1^4 a_ik_i$$

is left invariant it is evident that $r \cdot \Phi$ must vanish. That is

$$a_2 m_1 k_1 - a_1 m_1 k_2 + a_4 m_2 k_3 - a_3 m_2 k_4 = 0$$

which requires that all the a's vanish. However if we require that $r' = \lambda r$ then we have $r \cdot \Phi = \lambda r$ which leads to the solutions

$$a(k_1 \pm ik_2), \quad b(k_3 \pm ik_4).$$

Using these values for r we have

$$r_1' = (k_1 + ik_2) \cdot \Psi = k_1 + ik_2 + im_1(ik_1 + ik_2)\, dt$$

or

$$\frac{dr_1}{dt} = im_1(k_1 + ik_2).$$

$$r_2' = (k_1 - ik_2) \cdot \Psi = k_1 + ik_2 + im_1(k_1 - ik_2)\, dt$$

$$\frac{dr_2}{dt} = - im_1(k_1 - ik_2).$$

Likewise for the other two we have

$$\frac{dr_3}{dt} = im_2(k_3 + ik_4),$$

$$\frac{dr_4}{dt} = - im_2(k_3 - ik_4).$$

Thus the points on $k_1 + ik_2$ and $k_3 + ik_4$ progress by the factors im_1 and im_2 while the point on $k_1 - ik_2$ and $k_3 - ik_4$ progress by the factors $- im_1$ and $- im_2$.

If $m_1 = m_2$ it is easily seen that any vector of the pencils

$$(k_1 + ik_2) + \lambda(k_3 + ik_4), \quad (k_1 - ik_2) + \lambda(k_3 - ik_4)$$

is left invariant. These vectors lie in a plane in which every vector is a minimal vector. That is they lie in a plane which contains a generator of the imaginary sphere at infinity. Any plane cutting these two planes in a 1-vector will be left invariant since it will contain two invariant vectors. If $m_1 = -m_2$, the invariant pencils are

$$(k_1 + ik_2) + \lambda(k_3 - ik_4), \quad (k_1 - ik_2) + \mu(k_3 + ik_4).$$

Each of these pencils lies in a plane cutting the imaginary sphere at infinity in a generator and any plane cutting each of these planes in a vector will be left invariant. We will however discuss these planes from a different point of view.

The dyadic Ψ can be expressed in terms of plane coordinates by means of the double product.

$$(36)\qquad \Psi_2 = \tfrac{1}{2}\Psi{}^{\times}_{\times}\Psi = \tfrac{1}{2}(I_1 + \Phi dt){}^{\times}_{\times}(I_1 + \Phi dt) = \tfrac{1}{2}I_1{}^{\times}_{\times}I_1 + I{}^{\times}_{\times}\Phi dt$$
$$= I_2 + I{}^{\times}_{\times}\Phi dt$$

and the rotation is then expressed by the formula

$$C' = C\cdot(I_2 + I_1{}^{\times}_{\times}\Phi dt).$$

If we write $M = m_1k_{12} + m_2k_{34}$

$$\Phi = m_1(k_2k_1 - k_1k_2) + m_2(k_4k_3 - k_3k_4)$$
$$I{}^{\times}_{\times}\Phi = m_1(k_{23}k_{13} - k_{13}k_{23} + k_{24}k_{14} - k_{14}k_{24}) + m_2(k_{14}k_{13} - k_{13}k_{14} + k_{24}k_{23} - k_{23}k_{24})$$

If the complex 2-vector

$$C = \Sigma c_{ij}k_{ij}$$

is left invariant, that is if $C\cdot\Psi = C$

$$C\cdot(I_1{}^{\times}_{\times}\Phi) = m_1(c_{13}k_{23} - c_{23}k_{13} + c_{14}k_{24} - c_{24}k_{14}) + m_2(c_{13}k_{14} - c_{14}k_{13} + c_{23}k_{24} - c_{24}k_{23}) = 0.$$

From which we get

$$(37)\qquad m_1c_{13} = -m_2c_{24},\quad m_1c_{23} = m_2c_{14},\quad m_1c_{14} = m_2c_{23},\quad -m_1c_{24} = m_2c_{13}.$$

If $m_1 \neq \pm m_2$ these equations are satisfied only when

$$c_{13} = c_{24} = c_{14} = c_{23} = 0.$$

Thus any complex of the linear pencil

$$c_{12}k_{12} + c_{34}k_{34}$$

is left invariant. The only simple planes belonging to this pencil are k_{12} and k_{34}. Hence these are the only planes whose magnitude and position are left invariant by the rotation.

If $m_1 = \pm m_2$ equations (37) can evidently be satisfied if $c_{13} = \pm c_{24}$, $c_{14} = \pm c_{23}$. In this case the complex

$$C = c_{12}k_{12} + c_{13}(k_{13} \pm k_{42}) + c_{14}(k_{14} + k_{23}) + c_{34}k_{34}$$

is left invariant for all values of the coefficients. The planes of this system of complexes are determined by the relation

$$C\times C = c_{12}c_{34} \pm (c_{13}^2 + c_{14}^2)$$

which, if we put $c_{12} = 1$, gives for the invariant planes

$$(38)\quad P = k_{12} + c_{13}(k_{13} \pm k_{42}) + c_{14}(k_{14} + k_{23}) \pm (c_{13}^2 + c_{14}^2)k_{34}$$

Therefore, *If the rates of rotation in the planes* M_1 *and* M_2 *are different the only planes left invariant by* (33) *are* M_1 *and* M_2 *but if the rates of rotation in the two planes are the same or differ only in sign then a two parameter family of planes which belongs to a three parameter linear system of complexes is left invariant.*[12]

The planes of the system (38) all cut the planes

$$(k_{13} \pm k_{42}) + i(k_{14} \pm k_{23}),\ (k_{13} \pm k_{42}) - i(k_{14} \pm k_{23}).$$

These are the planes mentioned above.

We saw that in case $\mathrm{m}_1 = \pm m_2$ the complex M can be resolved in ∞^2 different ways into the sum of two completely perpendicular planes. The pairs of planes belong to the set (38). The transformation (34) can then be represented in ∞^2 different ways as the sum of rotations parallel to pairs of completely perpendicular planes. The rates of rotation parallel to both planes of a pair are the same but different for different pairs.

The above set of invariant planes were found under the condition that their magnitude be left unchanged. We might however have

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12 In the article referred to in note 2 Cole states the theorem "Every rotation in a four dimensional space for which $\vartheta \neq 0$. (The condition here would be that neither m_1 nor m_2 is zero) can be reduced to a succession of two simple rotations whose fixed planes are absolutely perpendicular to each other. This decomposition can be effected in only one way." From the above theorem it is evident that this statement is inaccurate. He discussed finite rotations and writes the equations of the rotation

$$x' = x\cos\theta - y\sin\theta,\quad y' = x\sin\theta + y\cos\theta$$
$$z' = z\cos\varphi - \omega\sin\varphi,\quad \omega' = z'\sin\varphi + \omega'\cos\varphi.$$

He states that the invariant planes of this rotation are the xy- and zw-planes. This is true of $\theta \neq \varphi$. But if $\theta = \varphi$ any line passing through 0 and lying in the planes $x + iy = 0$, $z + i\omega = 0$ is left invariant also any line passing through 0 and lying in the plane $x - iy = 0$, $z - i\omega = 0$ is left invariant. Hence any plane containing two of these invariant lines will be kept invariant. There is a two parameter family of these planes which are real and therefore ∞^2 real planes are left invariant. If $\theta = -\varphi$ then every line passing through 0 and lying in one of the planes $x + iy = 0$, $z - i\omega = 0$ or $x - iy = 0$, $z + i\omega = 0$ is left invariant and all the planes passing through 0 and cutting each of these planes in a line is left invariant. In the first case it is easy to see that the plane $x = z$, $y = \omega$ is left invariant and in the second case $x = \omega$, $y = z$ is left invariant. The error in Cole's work arose from the fact that in determining the coordinates of the invariant planes he failed to take into account that it was possible for all the denominators of his expressions to vanish simultaneously.

planes changed by the rotation into multiples of themselves. In this case

$$C \cdot (I_1 {}^{\times}_{\times} \Phi) = \lambda C$$

which leads to the relations

$$\begin{aligned} m_1 c_{13} - m_2 c_{24} &= \lambda c_{23}, \\ -m_1 c_{23} - m_2 c_{14} &= \lambda c_{13}, \\ m_1 c_{14} - m_2 c_{23} &= \lambda c_{24}, \\ -m_1 c_{24} + m_2 c_{13} &= \lambda c_{14}. \end{aligned}$$

Four values of λ render this system consistent and the corresponding invariant planes are

$$(k_{13} + k_{42}) \pm i(k_{14} + k_{23}), \; k_{13} - k_{42} \pm i(k_{14} - k_{23}).$$

No real plane satisfies this condition.

7. **Rotations in any even space**. Equation (5) is a rotation in a space of $2p$ dimensions if we consider M as a complex lying in that space. As before the dyadic representing the rotation, if we write M in terms of p mutually completely perpendicular unit planes which for convenience we will take as coordinate planes, is

$$\begin{aligned} \Psi &= I_1 + [m_1(k_2k_1 - k_1k_2) + m_2(k_4k_3 - k_3k_4) + \ldots + m_p(k_{2p}k_{2p-1} - k_{2p-1}k_{2p})]dt. \\ &= I_1 + \Phi dt. \end{aligned}$$

The same transformation expressed in plane coordinates is

$$\begin{aligned} \Psi_2 &= \tfrac{1}{2} I_1 {}^{\times}_{\times} I_1 + I_1 {}^{\times}_{\times} \Phi dt. \\ &= I_2 + I_1 {}^{\times}_{\times} \Phi dt \end{aligned}$$

where

$$\begin{aligned} I_1 &= \Sigma k_i k_i \\ I_2 &= \Sigma k_{ij} k_{ij}. \end{aligned}$$

The same argument used in the preceding section will show that the p mutually perpendicular planes into which M is resolved are all left invariant. If the m's are all distinct these are all the invariant planes, but if n of them are equal there are $\infty^{2(n-1)}$ invariant planes and the rotation can be resolved in an infinite number of ways into rotations parallel to p mutually perpendicular planes.

IV.

8. **Surfaces in 4-space left invariant by all the rotations having the same two fixed planes.** All the rotations represented by the equation

$$r' = r + M\cdot r\,dt = r + (m_1M_1 + m_2M_2)\cdot r\,dt \tag{39}$$

where M_1 and M_2 are unit planes and m_1 and m_2 are allowed to vary form a group since each transformation of the set leaves M_1 and M_2 invariant and consequently the product of two of them will leave these planes invariant also. The direction which a point will move by (39) with fixed values for m_1 and m_2 is

$$\frac{dr}{dt} = (m_1M_1 + m_2M_2)\cdot r = M\cdot r. \tag{40}$$

If m_1 and m_2 are allowed to vary it is seen at once that all the directions which a given point can take lie in a plane since they are linear functions of the two vectors $M_1\cdot r$ and $M_2\cdot r$. It is seen also from this equation that the ratio $m_1 : m_2$ is all that need be considered since their actual values are necessary for determining the magnitude of $\frac{dr}{dt}$ and not its direction. If we give m_1 and m_2 definite values (40) will be the differential equation of the path curve described by the end of the vector r by this particular rotation. The unit tangent to this curve is

$$\tau = \frac{dr}{ds} = M\cdot r\,\frac{dt}{ds}$$

Since the magnitude of r is unity we have

$$1 = \frac{dr}{ds}\cdot\frac{dr}{ds} = (M\cdot r)\cdot(M\cdot r)\left(\frac{dt}{ds}\right)^2$$
$$= [m_1^2(M_1\cdot r)^2 + m_2^2(M_2\cdot r)^2]\left(\frac{dt}{ds}\right)^2.$$

Hence

$$\left(\frac{ds}{dt}\right)^2 = m_1^2(M_1\cdot r)^2 + m_2^2(M_2\cdot r)^2. \tag{41}$$

As the transformation is a rotation r is a vector of constant length and also the projections on the fixed planes M_1 and M_2 are also the

same for all positions which r can take by the rotation. Then $(M_1\cdot r)\cdot(M_1\cdot r)$ is constant and equal to the square of the projection of r on the fixed plane M_1. Likewise $(M_2\cdot r)\cdot(M_2\cdot r)$ is equal to the square of the projection of r on the plane M_2. It follows at once then that ds is a constant since the expression in the bracket in (41) is constant.

The curvature of the path curve is

$$(42)\qquad \begin{aligned} C &= \frac{d\tau}{ds} = \frac{d^2r}{ds^2} = \left(M\cdot\frac{dr}{ds}\right)\frac{dt}{ds} = M\cdot(M\cdot r)\left(\frac{dt}{ds}\right)^2 \\ &= [m_1^2M_1\cdot(M_1\cdot r) + m_2^2M_2\cdot(M_2\cdot r)]\left(\frac{dt}{ds}\right)^2 \end{aligned}$$

The vectors $M_1\cdot(M_1\cdot r)$ and $M_2\cdot(M_2\cdot r)$ are the projections of r on the fixed planes M_1 and M_2 respectively and therefore constant in length. Hence: *The path curves are curves of constant scalar curvature.*

The unit vector in the direction of the curvature C is

$$c = \frac{M\cdot(M\cdot r)}{\sqrt{[M\cdot(M\cdot r)]^2}} = \frac{M\cdot(M\cdot r)}{\sqrt{C\cdot C}}$$

The vectors τ and c are unit vectors and perpendicular to each other Hence $\tau\times c$ will be the unit osculating plane to the path curve. The first torsion of the path curve is the rate of change of this plane with respect to the arc.

$$T = \frac{d}{ds}(\tau\times c) = \frac{d\tau}{ds}\times c + \tau\times\frac{dc}{ds} = \tau\times\frac{dc}{ds}$$

Since $\dfrac{d\tau}{ds} = c\sqrt{[M\cdot(M\cdot r)]^2}$ the product $\dfrac{d\tau}{ds}\times c = 0$. Substituting the values above we have

$$T = (M\cdot r)\times\{M\cdot[M\cdot(M\cdot r)]\}\left(\frac{dt}{ds}\right)^2\frac{1}{\sqrt{C\cdot C}}$$

$$\begin{aligned} = (m_1M_1 + m_2M_2)\times\{m_1^3M_1\cdot[M_1\cdot[M_1\cdot r)] \\ + m_2^3M_2\cdot[M_2\cdot(M_2\cdot r)]\left(\frac{dt}{ds}\right)^3\frac{1}{\sqrt{C\cdot C}} \end{aligned}$$

But since $M_1\cdot(M_1\cdot r)$ is the projection of r on M_1, $M_1\cdot[M_1\cdot(M_1\cdot r)]$ is a vector in M_1 perpendicular to this projection and consequently is equal to $M_1\cdot r$. Similarly for the second term. Then

$$T = m_1m_2(m_2^2 - m_1^2)\,(M_1\cdot r)\times(M_2\cdot r)$$

But since $M_1 \cdot r$ and $M_2 \cdot r$ are vectors of constant length and lie in perpendicular planes their cross product has constant magnitude. Therefore *the scalar first torsion of the path curves is constant.*

From the above expression for T we see that if $m_1 = 0$ or $m_2 = 0$ or $m_1 = \pm m_2$ the torsion vanishes, that is the plane $\tau \times c$ is independent of the point on the curve. *Therefore in the group of infinitesimal transformations which leave the same pair of perpendicular planes* M_1 *and* M_2 *invariant there are four transformations whose path curves are plane curves.* If $m_1 = 0$ the motion reduces to a rotation parallel to the plane M_2 and the path curve is the circle whose plane is parallel to M_1 and whose center is on M_1. Likewise if $m_2 = 0$ the path curve is a circle in a plane parallel to M_2 and whose center lies in M_1. If $m_1 = \pm m_2$ equation (42) becomes

$$C = m_1^2[M_1 \cdot (M_1 \cdot r) + M_2 \cdot (M_2 \cdot r)] \left(\frac{dt}{ds}\right)^2.$$

But since $M_1 \cdot (M_1 \cdot r)$ and $M_2 \cdot (M_2 \cdot r)$ are the projections of the vector r on the planes M_1 and M_2 respectively and these planes are perpendicular the expression in the brackets is evidently equal to r. Hence in this case

$$C = m_1^2 r \left(\frac{dt}{ds}\right)^2.$$

From (41) we see that in this case

$$\left(\frac{ds}{dt}\right)^2 = m_1^2(r \cdot r).$$

Then

$$C = \frac{r}{r \cdot r}$$

and hence

$$C \cdot C = \frac{r}{r \cdot r}.$$

That is the scalar of the curvature of the path curve is the reciprocal of the length of r. Then in this case the path curves are circles with center at the origin. The plane of the circle is $r \times (M_1 \cdot r + M_2 \cdot r)$. This plane changes only in magnitude if we change the length of r. Hence the plane is left invariant by the transformation $dr = m_1(M_1 + M_2) \cdot r \, dt$. That is through each point in space passes one plane left invariant by this transformation. From (39) the directions of the

path curves through a point, given by the transformations for which $m_1 = 0$, $m_2 = 0$, $m_1 = m_2$, $m_1 = -m_2$ form a harmonic pencil.

From (42) we see that if r is held fixed and m_1 and m_2 are allowed to vary the end of the curvature vector traces out the line joining the ends of the vectors $\frac{M_1\cdot(M_1\cdot r)}{(M_1\cdot r)^2}$ and $\frac{M_2\cdot(M_2\cdot r)}{(M_2\cdot r)^2}$ and it is seen that for real values of m_1 and m_2 the only points obtained are those on the segment joining the ends of these two vectors and that the ratios $\frac{m_1}{m_2}$ and $-\frac{m_1}{m_2}$ give the same point. Hence each point of the segment is counted twice. We will call this the *curvature segment*. Two directions $\frac{dr}{dt} = (m_1M_1 + m_2M_2)\cdot r$ and $\frac{dr'}{dt} = (m_1'M_1 + m_2'M_2)\cdot r$ are perpendicular if they satisfy the relation.

$$\frac{dr}{dt}\cdot\frac{dr'}{dt} = 0 = (m_1M_1\cdot r + m_2M_i\cdot r)\cdot(m_1'M_1\cdot r + m_2'M_2\cdot r)$$

$$= m_1m_1'(M_1\cdot r)^2 + m_2m_2'(M_2\cdot r)^2.$$

Hence

$$\frac{m_2'}{m_1'} = -\frac{(M_1\cdot r)^2}{(M_2\cdot r)^2}\frac{m_1}{m_2}.$$

Two perpendicular directions are then $m_1M_1\cdot r + m_2M_2\cdot r$ and $m_2(M_2\cdot r)^2(M_1\cdot r) + m_1(M_1\cdot r)^2(M_2\cdot r)$. The curvature for these two directions is

$$c_1 = [m_1^2M_i\cdot(M_1\cdot r) + m_2^2M_2\cdot(M_2\cdot r)]\left(\frac{dt}{ds}\right)_1^2$$

$$c_2 = [m_2^2[(M_2\cdot r)\cdot(M_2\cdot r)]^2M_1\cdot(M_1\cdot r) + m_1^2[(M_1\cdot r)\cdot(M_1\cdot r)]^2M_2\cdot(M_2\cdot r)]\left(\frac{dt}{ds}\right)_2$$

$$\left(\frac{dt}{ds}\right)_1 = \frac{1}{m_1^2(M_1\cdot r)^2 + m_2^2(M_2\cdot r)^2}$$

$$\left(\frac{dt}{ds}\right)_2 = \frac{1}{m_2^2[(M_2\cdot r)\cdot(M_2\cdot r)]^2(M_1\cdot r)^2 + m_1^2[(M_1\cdot r)\cdot(M_1\cdot r)]^2(M_2\cdot r)^2}$$

$$= \frac{1}{[(M_1\cdot r)^2(M_2\cdot r)^2]\,[m_1^2(M_1\cdot r)^2 + m_2^2(M_2\cdot r)^2}$$

Substituting these values of $\left(\frac{dt}{ds}\right)_1$ and $\left(\frac{dt}{ds}\right)_2$ in the expressions for c_1

and c_2 we have

$$c_1 + c_2 = \frac{M_1\cdot(M_1\cdot r)}{(M_1\cdot r)^2} + \frac{M_2\cdot(M_2\cdot r)}{(M_2\cdot r)^2} = 2\,h.$$

That is the sum of the curvature vectors for two perpendicular directions through the point is independent of the pair of directions taken and is equal to the sum of the curvatures of the path curves of the rotations $m_1 = 0$ and $m_2 = 0$ since $\dfrac{M_1\cdot(M_1\cdot r)}{(M_1\cdot r)^2}$ and $\dfrac{M_2\cdot(M_2\cdot r)}{(M_2\cdot r)^2}$ are the curvatures in these two directions. It is also to be noted that the directions for which the curvatures are the same are harmonically separated by the directions $m_1 = 0$, $m_2 = 0$. The directions which have curvature equal to h are perpendicular to each other and hence bisect the angle between $m_1 = 0$ and $m_2 = 0$.

Since the length of the radius of curvature is the reciprocal of the scalar curvature and its direction coincides with C the locus of the centers of curvature is the inverse of the curvature segment with respect to the unit circle with center at the extremity of r. Hence; *The locus of the centers of curvature of all the path curves of this group which pass through a given point is a circle of which the diameter is the line joining the origin to the point in question. For real directions through the point the centers of curvature lie on a quadrant of this circle.* From this it is evident that the curves with minimum curvature are in the directions $m_1 = m_2$ and $m_1 = -m_2$ and hence for these directions the curvature is perpendicular to the curvature segment. This can be seen also directly from (42). For the curvature of these curves being in the direction of r and the curvature segment being $\dfrac{M_1\cdot(M_1\cdot r)}{(M_1\cdot r)^2} - \dfrac{M_2\cdot(M_2\cdot r)}{(M_2\cdot r)^2}$ we have

$$r\cdot\left[\frac{M_1\cdot(M_1\cdot r)}{(M_1\cdot r)^2} - \frac{M_2\cdot(M_2\cdot r)}{(M_2\cdot r)^2}\right] = 0.$$

Since $M_1\cdot(M_1\cdot r)$ is the projection of r on M_1 and has the same length as $M_1\cdot r$ the product $r\cdot[M_1\cdot(M_1\cdot r)]$ is then the length of the projection of r on M_1 multiplied by the length of r. Hence the first term of the product reduces to unity and likewise for the second term and hence the whole product vanishes.

The curvature vectors of the path curves lie in a plane determined by $M_1\cdot(M_1\cdot r)$ and $M_2\cdot(M_2\cdot r)$. But $M_1\cdot r$ is a vector in M_1 perpendicular to r and $M_1\cdot(M_1\cdot r)$ is the projection of r *on* M_1 hence these

two vectors are perpendicular to each other. The vector $M_1 \cdot r$ is perpendicular to $M_2 \cdot (M_2 \cdot r)$ since they lie in completely perpendicular planes. Therefore $M_1 \cdot r$ is perpendicular to the plane of the curvature vector. Similarly $M_2 \cdot r$ is also perpendicular to this plane. Hence, *The path curves of a given point by the transformations of the group of rotations which leave the same two completely perpendicular planes fixed are all tangent to a plane A, and have constant curvature and torsion. The ends of the curvature vectors lie on a line cutting the two fixed planes. The plane in which these curvature vectors lie is perpendicular to the plane A. There are four of the path curves which are circles. The tangents to two of these are perpendicular to each other and the tangents to the four form a harmonic pencil. The centers of curvature lie on a circle whose diameter is the line joining 0 to the given point. The centers of curvature of the real curves lie on one quadrant of this circle.*

As the vector r is rotated by (39) the plane A will envelope a surface which will be left invariant by every transformation of the group. To obtain the equations of this surface we will integrate the vector differential equation (39). Let

$$r = x_1 k_1 + x_2 k_2 + x_3 k_3 + x_4 k_4)$$
$$M_1 = k_{12}, \qquad M_2 = k_{34}$$

Then (39) becomes

$$k_1 dk_1 + k_2 dk_2 + k_3 dk_3 + k_4 dx_4 = (m_1 k_{12} + m_2 k_{34}) \cdot (x_1 k_1 + x_2 k_2 + x_3 k_3 + x_4 k_4) dt$$
$$= (m_1 x_2 k_1 - m_1 x_1 k_2 + m_2 x_4 k_3 - m_2 x_3 k_4) dt$$

which is equivalent to the set of differential equations

$$\frac{dx_1}{dt} = m_1 x_2, \quad \frac{dx_2}{dt} = -m_1 x_1,$$
$$\frac{dx_3}{dt} = m_2 x_4, \quad \frac{dx_4}{dt} = -m_2 x_3.$$

Dividing and integrating we obtain for the first integrals

$$x_1^2 + x_2^2 = a^2, \quad x_3^2 + x_4^2 = b^2. \tag{43}$$

The constants a and b are so determined that the curve will pass through the initial point. These then are the equations of the surface left invariant by each transformation of the group. The planes A are the tangent planes to the surface and the normal planes are those in which the curvature vectors lie. These normal planes all pass through

the origin, that is through the point of intersection of the fixed planes. The curvature of a geodesic always lies in the normal plane to the surface from which we can conclude that the path curves are geodesics of the surface (43). We will however show this directly. The parametric equations of the surface are

$$(44) \qquad \begin{aligned} x_1 &= a \cos u, \quad x_2 = a \sin u, \\ x_3 &= b \cos v, \quad x_4 = b \sin v. \end{aligned}$$

Then

$$ds^2 = a^2du^2 + b^2dv^2$$

This shows that the surface is developable.[13] The geodesics are then the lines given by the relation

$$v = Au + B$$

which substituted in the differential equations of the path curves we find they are satisfied provided $A = \frac{m_2}{m_1}$. Hence the path curves are the geodesics. Through each point of the surface passes four geodesics which are circles. The planes of two of them are completely perpendicular being parallel respectively to the two fixed planes. The other two circular geodesics make equal angles with the two preceding, and have their centers at the origin. Their planes consequently intersect in a line, that is, lie in a 3-space. The surface can be generated by rotating any one of the path curves by any transformation of the group. Therefore it can be generated by moving a circle of fixed radius and plane parallel to the x_3x_4-plane with center in the x_1x_2-plane so that it always cuts a fixed circle lying in a plane parallel to the x_1x_2-plane. It can also be generated by a circle of radius $\sqrt{a^2 + b^2}$ with center at 0 which always cuts a fixed circle of radius $\sqrt{a^2 + b^2}$ and center at 0. The plane of the variable circle is inclined at a fixed angle to the plane of the fixed circle.

Equations (43) show that the surface is of order four. Therefore a 3-space which cuts it in a circle must cut it again in a curve of order two. Consider first a 3-space which contains one of the circles with center at 0. This will also contain a second one of these circles since they lie by twos in all the 3-spaces containing one of them. If then we pass a sphere through one of these circles for which it is a great circle, it will contain a second one of them which will also be a great circle. Hence such a sphere is tangent to the surface at two diametrically opposite points.

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[13] See Levi, *loc. cit.*

The plane of one of these circles is obtained by letting $v = u + \theta$ in equations (44). The equation of this path curve then becomes

$$(45) \qquad \begin{aligned} x_1 &= a \cos u, & x_2 &= a \sin u \\ x_3 &= b \cos (u + \theta), & x_4 &= b \sin (u + \theta) \end{aligned}$$

and the plane of the curve becomes

$$(46) \qquad \begin{aligned} x_3 &= \frac{b}{a} \cos \theta\, x_1 - \frac{b}{a} \sin \theta x_2, \\ x_4 &= \frac{b}{a} \cos \theta\, x_2 + \frac{b}{a} \sin \theta x_1. \end{aligned}$$

Varying θ we obtain all circles which form one generation of the surface. These circles have no point in common. The second generation can be obtained by putting $v = -u + \theta$. The equations of these circles then is

$$(47) \qquad \begin{aligned} x_1 &= a \cos u, & x_2 &= a \sin u \\ x_3 &= b \cos(-u + \theta), & x_4 &= b \sin (-u + \theta). \end{aligned}$$

The circles of (47) do not intersect each other but each one of (47) intersects each one of (45) in two diametrically opposite points. The points of intersection are $u = \dfrac{\varphi - \theta}{2}$, $u = \dfrac{\varphi + \theta + 2\pi}{2}$. If these circles are used as parameter curves the equation of the surface becomes

$$\begin{aligned} x_1 &= a \cos (u + v), & x_2 &= a \sin (u + v), \\ x_3 &= b \cos (u - v), & x_4 &= b \sin (u - v). \end{aligned}$$

From (46) we see that the locus of the planes of these circles is

$$\frac{x_1^2 + x_2^2}{a^2} = \frac{x_3^2 + x_4^2}{b^2}.$$

In fact this quadric cone contains both sets of planes.

The planes of the other two generations of circles lie on the cylinders

$$x_1^2 + x_2^2 = a^2, \qquad x_3^2 + x_4^2 = b^2.$$

The planes on one of these cylinders are parallel to each other and consequently two of them determine a 3-space, that is, the 3-space which passes through one of these planes will contain another of the same cylinder. Then in this second double generation of circles,

circles of the same generation intersect in two points while those of opposite generations intersect in one point. Equations (47) show that the surface is also a translation surface.

Wilson and Moore [14] discussed the locus of the end of the normal curvature vector (the indicatrix) of curves passing through a given point of a surface and found that in general it is a conic. But when this indicatrix becomes a linear segment the surface has some properties of surface in 3-space. On such a surface lines of curvature can be defined as in 3-space and will be orthogonal. If we define lines of curvature as lines of maximum or minimum normal curvature we find in general there are four directions through each point but in 3-space these four directions divide into two sets of two, one the asymptotic lines and the other the lines of curvature. For surfaces whose indicatrix reduces to a linear segment not passing through the surface point in question these four directions of maximum and minimum radii of curvature again factor into two sets; one giving the curves called by Segre characteristics and the other giving lines analogous to lines of curvature in 3-dimensions. For the surface here considered all four sets of these curves are circles.

We saw that the curvature segment or indicatrix cut the planes M_1 and M_2 in the ends of the projection of r on these planes. Then as r is rotated the curvature segment will cut the circles generated by these projections. Hence the locus of the curvature segment will be the congruence of lines cutting two given circle. Also the mean curvature defined by the curvatures of two orthogonal directions.

$$2h = c_1 + c_2$$

is the vector from the surface point to the middle of the curvature segment. Then the locus of the end of the mean curvature vector will be the surface

$$x_1^2 + x_2^2 = \left(\frac{a^2 - 1}{a^2}\right)^2$$

$$x_3^2 + x_4^2 = \left(\frac{b^2 - 1}{b^2}\right)^2$$

which is a surface like (43).

We have here considered general positions of the vector r but an interesting case arises when r is so located that its projection on M_1

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is equal in length to its projection on M_2. On the surface generated by the path curves of the group the circles for which $m_1 = m_2$ and $m_1 = -m_2$ are orthogonal. Hence the directions of the circular sections form two orthogonal pairs. The center of mean curvature bisects the curvature segment.

Rotations in 5-space give nothing new since the path curves will lie in the 4-space perpendicular to the fixed axis of the rotation and passing through the given point.

9. **Rotations in 6-space leaving the same three mutually perpendicular planes, invariant.** We will next consider the case of 6-space in detail before generalizing. Evidently these transformations form a group. The directions which a point can move by the various transformations of the group form a 3-space. These directions are defined by

$$\tau = \frac{dr}{ds} = (m_1M_1 + m_2M_2 + m_3M_3)\cdot\frac{rdt}{ds} \tag{48}$$

τ is then a unit vector tangent to the curve given by a particular set of values of m_1, m_2, m_3. Squaring (48) we get

$$\left(\frac{ds}{dt}\right)^2 = m_1^2(M_1\cdot r)^2 + m_2^2(M_2\cdot r)^2 + m_3^2(M_3\cdot r)^2 \tag{49}$$

which is constant for given values of m_1, m_2, m_3 since $(M_1\cdot r)$, $(M_2\cdot r)$, $(M_3\cdot r)$ are of constant length. The curvature of the path curves is

$$C = \frac{m_1^2M_1\cdot(M_1\cdot r) + m_2^2M_2\cdot(M_2\cdot r) + m_3^2M_3\cdot(M_3\cdot r)}{m_1^2(M_1\cdot r)^2 + m_2^2(M_2\cdot r)^2 + m_3^2(M_3\cdot r)^2}. \tag{50}$$

This shows that the end of the curvature vectors lie in a plane determined by the projections of r on the three planes M_1, M_2, M_3. We see that for real values of m_1, m_2, m_3 that is for real directions through the given point, the points lie inside this triangle, which we will call the *curvature triangle*. To each point in the curvature triangle corresponds four sets of values of m_1, m_2, m_3, that is, four curves through the point. Each of these curves have the same curvature.

The angle between two directions $\frac{dr}{ds}$ and $\frac{dr'}{ds}$ is given by the formula

$$\frac{dr}{ds}\cdot\frac{dr'}{ds} = \frac{m_1m_1^1(M_1\cdot r)^2 + m_2m_2^1(M_2\cdot r)^2 + m_3m_3^1(M_3\cdot r)^2}{\sqrt{m_1^2(M_1\cdot r)^2 + m_2^2(M_2\cdot r)^2 + m_3^2(M_3\cdot r)^2}\sqrt{m_1^{12}(M_1\cdot r)^2 + m_2^{12}(M_2\cdot r)^2 + m_3^{12}(M_3\cdot r)^2}} \tag{51}$$

The condition that the two directions be orthogonal is

$$(52)\qquad m_1m_1^1(M_1\cdot r)^2 + m_2m_2^1(M_2\cdot r)^2 + m_3m_3^1(M_3\cdot r)^2 + 0.$$

This defines a plane of directions perpendicular to m_1^1, m_2^1, m_3^1.

From (48) we see that a linear relation among the m's gives a plane of directions through the point and from (50) we see that the end of the curvature vector will describe a conic in the curvature triangle determined by the points, $\dfrac{M_1\cdot(M_1\cdot r)}{(M_1\cdot r)^2}$, $\dfrac{M_2\cdot(M_2\cdot r)}{(M_2\cdot r)^2}$, $\dfrac{M_3\cdot(M_3\cdot r)}{(M_3\cdot r)^2}$ since the substitution of (52) in (50) gives a quadratic relation in m_1, m_2, m_3. To simplify the work let

$$\frac{M_1\cdot(M_1\cdot r)}{(M_1\cdot r)^2} = x,\quad \frac{M_2\cdot(M_2\cdot r)}{(M_2\cdot r)^2} = y,\quad \frac{M_3\cdot(M_3\cdot r)}{(M_3\cdot r)^2} = z$$

$$m_1^2(M_1\cdot r)^2 = \lambda^2,\quad m_2^2(M_2\cdot r)^2 = \mu^2,\quad m_3^2(M_3\cdot r)^2 = \nu^2$$

Then (50) takes the form

$$(53)\qquad C = \frac{\lambda^2x + \mu^2y + \nu^2z}{\lambda^2 + \mu^2 + \nu^2}.$$

Then a linear relation

$$(54)\qquad a\cdot\lambda + b\mu + c\nu = 0$$

is equivalent to saying that the direction λ, μ, ν is perpendicular to the direction defined by

$$(55)\qquad \frac{a}{\sqrt{(M_1\cdot r)^2}},\quad \frac{b}{\sqrt{(M_2\cdot r)^2}},\quad \frac{c}{\sqrt{(M_3\cdot r)^2}}.$$

From (53) and (54) the curvature of the directions perpendicular to (55) is

$$(56)\qquad C = \frac{c^2(\lambda^2x + \mu^2y) + (a\lambda + b\mu)^2z}{c^2(\lambda^2 + \mu^2) + (a\lambda + b\mu)^2}$$

This is a conic and as seen before it must lie inside the triangle determined by x, y, z and must therefore be an ellipse. The sides of the triangle are

$$(57)\qquad r_1 = \frac{\lambda^2x + \mu^2y}{\lambda^2 + \mu^2},\quad r_2 = \frac{\lambda^2x + \nu^2z}{\lambda^2 + \nu^2},\quad r_3 = \frac{\mu^2y + \nu^2z}{\mu^2 + \nu^2}.$$

The intersection of the conic (56) with r_1 is given by $a\lambda + b\mu = 0$ and hence the conic is tangent to r_1. The same argument shows that it is also tangent to r_2 and r_3. Hence: *The locus of the ends of the curvature vectors of curves perpendicular to a given direction is a conic tangent to the three sides of the triangle determined by the vectors x, y, z.*

The center of the conic (56) can be determined from the middle of the segment into which the conic projects on the x-, y- and z-axes. One end of each segment is at the origin. The projection on OX is

$$r = \frac{c^2\lambda^2 x}{c^2(\lambda^2 + \mu^2) + (a\lambda + b\mu)^2}.$$

The value of $\frac{\lambda}{\mu}$ which makes the denominator a minimum will make r a maximum. Similarly we can determine the projection of the ellipse on OY and OZ. Hence the center of the ellipse is the end of the vector

$$\rho_1 = \frac{(b^2 + c^2)x + (a^2 + c^2)y + (a^2 + b^2)z}{2(a^2 + b^2 + c^2)}$$

The point of the curvature triangle corresponding to the direction $a/\sqrt{(M_1 \cdot r)^2}$, $b/\sqrt{(M_2 \cdot r)^2}$, $c/\sqrt{(M_3 \cdot r)^2}$ is

$$\rho_2 = \frac{a^2x + b^2y + c^2z}{a^2 + b^2 + c^2}.$$

From which we have

$$\frac{2\rho_1 + \rho_2}{3} = \frac{x + y + z}{3}.$$

The right side of this last equation is the median center of the curvature triangle. Hence the center of the curvature triangle is a point of trisection of the line joining the center of the conic to the point of the triangle corresponding to the direction $a/\sqrt{(M_1 \cdot r)^2}$, $b/\sqrt{(M_2 \cdot r)^2}$, $c/\sqrt{(M_3 \cdot r)^2}$. It can be shown that two perpendicular directions among those satisfied by (54) correspond to points at opposite ends of a diameter of the conic (56). Hence the points of the curvature triangle which correspond to three mutually perpendicular directions through the point form a triangle whose median center coincides with the median center of the curvature triangle. The points of the conic

will correspond to four different planes of directions through the point. These are given by the linear relations

$$\begin{aligned} a\lambda + b\mu + c\nu &= 0, \\ a\lambda + b\mu - c\nu &= 0, \\ a\lambda - b\mu + c\nu &= 0, \\ a\lambda - b\mu - c\nu &= 0. \end{aligned}$$

By substituting these four relations in (53) it is seen that we obtain the same conic. The same point in the curvature triangle correspond to the perpendicular direction for each relation. If $m_1 = 0$ or $m_2 = 0$ or $m_3 = 0$, the corresponding points in the triangle are on the sides of it. The rotations in this case are of the four dimensional type previously discussed.

The unit osculating plane is again the cross product of the unit tangent and the unit curvature. The unit curvature is

$$C = \frac{m_1^2 M_1\cdot(M_1\cdot r) + m_2^2 M_2\cdot(M_2\cdot r) + m_3^2 M_3\cdot(M_3\cdot r)}{m_1^2(M_1\cdot r)^2 + m_2^2(M_2\cdot r)^2 + m_3^2(M_3\cdot r)^2}.$$

The rate of change of the osculating plane $(\tau\times c)$ with respect to the arc is again the first torsion.

$$\begin{aligned} \tau = \frac{d}{ds}(\tau\times c) = \tau\times\frac{dc}{ds} &= (M\cdot r)\times\{M\cdot[M\cdot(M\cdot r)]\}\left(\frac{dt}{ds}\right)^3\frac{1}{\sqrt{C\cdot C}} \\ &= (m_1M_1\cdot r + m_2M_2\cdot r + m_3M_3\cdot r)\times\{\Sigma m_i^3 M_i \\ &\qquad \cdot[M_i\cdot(M_i\cdot r)]\}\left(\frac{dt}{ds}\right)^3\frac{1}{\sqrt{C\cdot C}} \end{aligned}$$

$M_i\cdot(M_i\cdot r)$ is the projection of r on M_i and $M_i\cdot[M_i\cdot(M_i\cdot r)]$ is a vector of equal length in M_i and perpendicular to $M_i\cdot(M_i\cdot r)$ and is therefore equal to $(M_i\cdot r)$. Hence we can write for the torsion

$$\begin{aligned} \tau &= (\Sigma m_iM_i\cdot r)\times\{\Sigma m_i^3(M_i\cdot r)\}\left(\frac{dt}{ds}\right)^2\frac{1}{\sqrt{C\cdot C}} \\ &= m_1m_2(m_1^2 - m_2^2)\,(M_1\cdot r)\times(M_2\cdot r) + m_1m_3(m_1^2 - m_3^2)\,(M_1\cdot r)\times(M_3\cdot r) \\ &\qquad + m_2m_3(m_2^2 - m_3^2)\,(M_2\cdot r)\times(M_3\cdot r)\left(\frac{dt}{ds}\right)^3\frac{1}{\sqrt{C\cdot C}}. \end{aligned}$$

For given values of m_1, m_2, m_3 the magnitude of this vector is constant. The path curves are then curves for which the rate of change of the unit osculating plane with respect to the arc is a vector of constant

magnitude. The vector τ will vanish if $m_i = m_j = 0$, $i \neq j$ or $m_i = 0$, $m_j = m_k$ $(i \neq j \neq k)$, or if $m_1 = \pm m_2 = \pm m_3$. The first case gives the transformations which leave two of the planes absolutely fixed and the path curves are circles whose center is the projection of the end of r on the absolutely fixed 4-space determined by the two absolutely fixed planes. The second correspond to the rotations leaving one of the planes absolutely fixed and the path curves are circles with center on the fixed plane and radius equal to the length of the perpendicular dropped from the end of r to the fixed plane. The curvature of the path curves for the last case is

$$C = \frac{M_1\cdot(M_1\cdot r) + M_2\cdot(M_2\cdot r) + M_3\cdot(M_3\cdot r)}{(M_1\cdot r)^2 + (M_2\cdot r)^2 + (M_3\cdot r)^2} =$$

$$\frac{r}{(M_1\cdot r)^2 + (M_2\cdot r)^2 + (M_3\cdot r)^2} = \frac{r}{\sqrt{r\cdot r}}$$

since $M_i\cdot(M_i\cdot r)$ is the projection of r on M_i and the sum of the projections of r on three mutually perpendicular planes is equal to r. Also from the definition of the dot product it is evident that the magnitudes of $M_i\cdot r$ and $M_i\cdot(M_i\cdot r)$ are equal. Hence these curves have curvature directed through the origin and are circles with center at the origin. The point on the curvature triangle corresponding to the direction of the tangents to these circles is the end of the vector

$$C = \frac{M_1\cdot(M_1\cdot r) + M_2\cdot(M_2\cdot r) + M_3\cdot(M_3\cdot r)}{(M_1\cdot r)^2 + (M_2\cdot r)^2 + (M_3\cdot r)^2}.$$

This vector is perpendicular to the plane of the curvature triangle. For two sides of the triangle are

$$\frac{M_1\cdot(M_1\cdot r)}{(M_1\cdot r)^2} - \frac{M_2\cdot(M_2\cdot r)}{(M_2\cdot r)^2}, \quad \frac{M_1\cdot(M_1\cdot r)}{(M_1\cdot r)^2} - \frac{M_3\cdot(M_3\cdot r)}{(M_3\cdot r)^2}$$

and it is seen at once that the dot product of C with either of these vectors vanishes and hence C is perpendicular to the plane determined by these two vectors. The four directions $m_1 = \pm m_2 = \pm m_3$ correspond to the same point in the curvature triangle viz. the foot of the perpendicular dropped from the end of r on the plane of this triangle. These circles are then the path curves of minimum curvature. The radius of curvature being the reciprocal of the curvature; *The locus of the centers of curvature of all path curves which pass through a given point is the inverse of the curvature triangle with respect to a*

unit sphere with center at the point in question and therefore is a sphere with r *for a diameter. The centers of curvature of the real curves lie on an octant of this sphere.* In four dimensions we found that the path curves corresponding to the center of the curvature segment were orthogonal but here the path curves corresponding to the median center of the curvature triangle do not have this property.

We saw that the locus of the end of the curvature vector for directions through a given point which satisfy a linear relation, that is curves tangent to the same plane, was a conic. This conic may degenerate into the sides of the curvature triangle counted twice. The directions corresponding to the points of one of these segments are all perpendicular to the direction corresponding to the opposite vertex. To a general point in the curvature triangle correspond four directions through the point but to a general point on one of the sides correspond two directions through the point and to a vertex of the triangle corresponds just one direction. A line in the plane of the curvature triangle is defined by the linear relation $\Sigma a_i m_i^2 = 0$, and this substituted in (48) shows that the corresponding directions through the point generate a quadric cone. In particular if one of the coefficients, a_1 say, is zero and the other two have opposite signs then the quadratic relation factors into two linear relations, each of which corresponds to a plane of directions through the point. From which we see that a linear relation involving only two of the m's gives a plane of directions whose curvature segment passes through a vertex of the curvature triangle. Two perpendicular directions correspond to the ends of the segment and from the fundamental configuration for the curvature of three mutually perpendicular directions it is at once seen that the curvature of the path curve perpendicular to this plane of directions will cut a side of the curvature triangle. The configuration can be shown by a simple figure. Let ABC be the curvature triangle, AD is the curvature segment corresponding to a plane of directions through the point depending on but two of the m's. Let H be the median center of the curvature triangle and G the middle of the curvature segment. The point corresponding to the direction perpendicular to the given plane i.e. to the directions corresponding to the segment AD must be such that the center of the triangle ADF is H. It is at once evident that F must be on BC and such that $DE = EF$. This together with $a_2 = 0$ or $a_3 = 0$ are the only cases in which $\Sigma a_i m_i^2 = 0$ can be factored into two linear relations. If D coincides with E, G will coincide with H.

A line which does not pass through a vertex of ABC will contain an

infinite number of point pairs corresponding to perpendicular directions. For, any point on the line can be taken as the center of a curvature ellipse which touches the three sides of ABC. The two points in which these ellipses cut the line correspond to perpendicular directions. The points corresponding to the directions perpendicular to these pairs will all lie on a line parallel to the given line.

Proceeding as in 4-space the differential equations of the path curves are found to be

$$(58) \quad \frac{dx_1}{dt} = m_1x_2, \quad \frac{dx_2}{dt} = -m_1x_1, \quad \ldots, \quad \frac{dx_5}{dt} = m_3x_6, \quad \frac{dx_6}{dt} = -m_3x_5.$$

The path curves will then all lie in the variety V_3^8 of order 8

$$(59) \qquad x_1^2 + x_2^2 = a^2, \; x_3^2 + x_4^2 = b^2, \; x_5^2 + x_6^2 = c^2,$$

where a, b, c are determined so that the curves all pass through the given point. If $m_i = \pm m_j$ the resulting path curves will lie in a 4-space but if $m_i = km_j$ ($k \neq \pm 1$) this is not the case. If $m_1 = \pm m_2 = \pm m_3$ the resulting path curves are plane curves. The argument is the same as that given in 4-space.

The parametric equations of V_3^8 are

$$x_1 = a\cos u, \; x_2 = a\sin u,$$
$$x_3 = b\cos v, \; x_4 = b\sin v,$$
$$x_5 = c\cos w, \; x_6 = c\sin w.$$

The element of arc is

$$(60) \qquad ds^2 = a^2du^2 + b^2dv^2 + c^2du^2$$

The variety can therefore be developed on a plane 3-space. The path curves have curvature lying in the normal 3-space (the 3-space determined by the surface point and the curvature triangle) and are therefore geodesics. That they are geodesics can be shown directly from the above equations as was done for rotations in 4-space. A linear relation among the m's will give a surface which is left invariant by a one parameter family of rotations. This is then a geodesic surface of the variety V_3^8. Furthermore the normal 4-space to this surface must contain the normal 3-space to V_3^8 and the ends of the curvature vectors of the pencil of geodesics passing through a given point will trace out a conic lying in the curvature triangle and since this lies in the normal to the surface these path curves must be geodesics on the surface K. A linear relation in the m's means a linear

relation in u, v, w. This substituted in (60) shows that the surface K is also developable. This surface differs from that studied in 4-space since for this one the indicatrix is a true ellipse and not a linear segment counted twice. The plane of the indicatrix does not pass through the surface point. In particular the linear relation $m_i = 0$ will lead to a geodesic surface all of whose geodesics are curves lying in a 4-space. This will cut K in a geodesic. Hence passing through each point of K pass three geodesics which lie in a 4-space.

For each point on V_3^8 there is a curvature triangle. The locus of these triangles consists of the planes cutting three fixed circles, one lying in each of the planes M_1, M_2, M_3.

We have found plane curves (circle) and curves lying in a 4-space which are left invariant, that is, path curves. It is evident that if a space curve is left invariant the space in which it lies must be left invariant. We saw that no 3-spaces were left invariant hence there are no 3-space path curves. Also we saw that the only 4-spaces left invariant were $M_1{\times}M_2$, $M_1{\times}M_3$, $M_2{\times}M_3$ hence the 4-space path curves mentioned above are the only ones that exist.

10. **Rotations in space of 2p dimensions which leave the same set of p mutually perpendicular planes invariant.** Having considered the case of four and six dimensions we can now easily generalize the results for space of $2p$ dimensions. Let the rotation be expressed in terms of the unit invariant planes

$$r' = r + \sum_{1}^{p} m_i M_i \cdot r \, dt \tag{61}$$

or

$$\tau = \frac{dr}{ds} = \sum_{1}^{p} m_i M_i \cdot r \frac{dt}{ds}.$$

The ∞^p transformations obtained if m_i vary, form a group and the different directions which a point can take by the various transformations of the group lie in a linear p-space. The curvature of the path curves at the point P is given by the formula

$$C = \frac{\Sigma m_1^2 M_i \cdot (M_i \cdot r)}{\Sigma m_i^2 (M_i \cdot r)^2}. \tag{62}$$

For given values of m_i the length of this vector curvature is seen to be independent of the position which r can take by the given rotation. That is the path curves are curves of constant curvature. These curvature vectors generate a p-space which is completely perpendicular

to the p-space generated by the tangents to the curves at the point in question. If any of the m's vanish the resulting rotation is equivalent to a rotation in a space of lower dimensions and therefore we shall assume that none of the m's vanish.

Equation (62) shows that, for real values of m_i, the end of the curvature vector will lie inside a p-point A, called the curvature p-point determined by the p points in which the extremity of the vector r projects on the p planes M_i. Each point of A will correspond to 2^{p-1} directions through P. The points in A which correspond to p mutually perpendicular directions through P for a p-point whose center of gravity coincides with the center of gravity of A. If a linear relation $\Sigma a_i m_i = 0$ exists among the m's (62) shows that the end of the corresponding curvature vectors will lie on a closed quadric in p–1 dimensions which touches the faces of A. The foot of the perpendicular dropped from the point P on the space in which A lies corresponds to the directions on the surface satisfying the relations

$$m_1 = \pm m_2 = \pm m_3 = \ldots . = \pm m_p \tag{63}$$

These curves then, 2^{p-1} in number, are curves of minimum curvature.

The first torsion of the path curves is given by the formula

$$T = \Sigma m_i m_j (m_i^2 - m_j^2)(M_i \cdot r) \times (M_j \cdot r) \left(\frac{dt}{ds}\right)^3 \frac{1}{\sqrt{C \cdot C}}.$$

This formula shows that the curves of zero torsion, excluding those corresponding to rotations in a space of less than 2 p dimensions, are those which satisfy relations (63). Hence these curves are plane curves, that is, circles. It is easy to show that the center of these curves is at the origin or at the intersection of the p invariant planes M_i. Other path curves are circles but these belong to rotations which leave one or more of the invariant planes absolutely fixed, that is, are equivalent to rotations in a lower space.

The differential equations of the path curves are

$$\frac{dx_1}{dt} = m_1 x_2, \quad \frac{dx_2}{dt} = - m_1 x_1, \quad \frac{dx_3}{dt} = m_2 x_4, \quad \frac{dx_4}{dt} = - m_2 x_3, \ldots .$$

One set of integrals of these equations is

$$x_1^2 + x_2^2 = a_1^2, \; x_3^2 + x_4^2 = a_2^2 \ldots . x_{2p-1}^2 + x_{2p}^2 = a_p^2$$

and therefore the path curves all lie in a variety V_p of order 2^p. The parameters equations of this variety are

$$\begin{aligned} x_1 &= a_1\cos u_1, \ x_2 = a_1\sin u_1, \\ x_3 &= a_2\cos u_2, \ x_4 = a_2\sin u_2, \\ &\cdots\cdots\cdots\cdots\cdots \\ x_{2p-1} &= a_p\cos u_p, \ x_{2p} = a_p\sin u_p. \end{aligned}$$

The element of arc then is

$$ds^2 = a_1^2du_1^2 + a_2^2du_2^2 + \ldots . a_p^2du_p^2$$

which shows that V_p can be developed on a plane space of p dimensions. A linear relation among the m's gives a variety of p–1 dimensions immersed in V_p and it is easily shown that this is also developable. The path curves are geodesics on both varieties.

For each point of V_p there is a curvature p-point and these p-points all cut p fixed circles, one lying in each of the invariant planes M_i.